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# Quasilinear approximation for the spectrum of wind-generated water waves

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According to Miles' theory of wind-wave generation, water waves grow if the curvature of the wind profile at the critical height is negative. As a result, the wind profile changes in time owing to the transfer of energy to the waves. In the quasilinear approximation (where the interaction of the waves with one another is neglected) equations for the coupled air–water system are obtained by means of a multiple-time-scale analysis. In this way the validity of Miles' calculations is extended, thereby allowing a study of the large-time behaviour.

While the water waves grow owing to the energy transfer from the air flow, the waves in turn modify the flow in such a way that for large times the curvature of the velocity profile vanishes. The amplitude of the waves is then limited because the energy transfer is quenched.

In the high-frequency range the asymptotic wave spectrum is given by a '$-4$' law in the frequency domain rather than the 'classical' '$-5$' law.

## 1. Introduction

A possible mechanism for the generation of water waves by the wind is resonant interaction of the gravity waves with a plane-parallel flow (Miles 1957). Resonance occurs at a critical height $z_c$ if $U(z_c) = C(\sigma)$, where $U$ is the air velocity and $C(\sigma)$ is the phase velocity of a wave with frequency $\sigma$. Only those waves grow for which the curvature of the velocity profile at the critical height is negative. This type of interaction has also been studied in other fields of physics. Some time ago Landau (1946) investigated the linear interaction of plasma waves with particles: the rate of change of the energy of the plasma waves is proportional to the derivative of the particle-velocity distribution function at the point of resonance (cf. Fabrikant 1976). For a plane-parallel flow this change of energy is proportional to the curvature of the velocity profile at the critical height. Also, in a plane-parallel flow near the point of resonance a pattern of closed streamlines is found, similar to Kelvin's 'cat's-eye' pattern, and the same feature occurs in the phase-space orbits of trapped particles in a given monochromatic plasma wave.

The linear theories of resonant interaction of gravity waves with a flow and plasma waves with particles are only valid on a short time scale, since in the course of time nonlinear effects may become important owing to the growth of the waves. Nonlinear effects on the interaction of plasma waves and particles have been studied by Vedenov, Velikhov & Sagdeev (1961) and Drummond & Pines (1962). One of the main results of these investigations was that plasma waves modify the particle distribution function, which leads eventually to saturation of the plasma waves. Davidson (1972) extended these results to include three-wave interactions and nonlinear wave–particle interactions. The plasma waves were assumed to have a sufficiently broad spectrum such that the random phase approximation is valid. The random-phase approximation or quasi-normal approach was also assumed to be valid by Hasselmann (1967) in his study of, for example, wave–wave interactions of water waves.

In this paper we are concerned with the effects of the water waves on the air flow. By means of the multiple-time-scale method we derive dynamic equations for the slowly varying energy density of the water waves and the wind velocity $U$ (§ 2). The growth of the water waves due to atmospheric input occurs on a long time scale since this energy transfer is proportional to the ratio of air density to water density. Hence, we have at least two time scales, namely one related to the relatively rapid water oscillations and one of the order of the energy transfer time from air to the water waves. Another reason for the use of the multiple-time-scale method is that an iterative solution of a set of nonlinear equations (in this case the Euler equations plus boundary conditions) usually gives rise to secular terms in time. The introduction of different time scales then provides freedom to prevent secularity. The condition resulting from the elimination of secularity gives the equations for the slow time dependence of the wave energy and the air speed.

We confine ourselves to a discussion of the quasilinear approximation, i.e. the effect of (four) wave–wave interactions and energy dissipation due to wave breaking on the evolution of the energy density is neglected.

In addition, the effect of air turbulence on the velocity profile is neglected. Although these approximations are probably not justified, the purpose of this paper is to discuss the coupling of waves and wind only in order to see what typical features for such a system may be found.

Section 3 is devoted to the study of some exact consequences of the quasilinear theory of wind-generated water waves. The coupled air–water system tends to an asymptotic equilibrium for which the wind profile becomes linear, corresponding to a frequency spectrum that shows an $f^{-4}$ dependence on frequency in the high-frequency range. Phillips (1958), on the other hand, obtained an $f^{-5}$ law on dimensional grounds. We must note however, that the $f^{-4}$ law stems from a different physical mechanism. While the $f^{-5}$ law is based on the balance of energy input from the air, transfer of energy due to nonlinear interaction and dissipation of energy through wave-breaking (Hasselmann 1974), the $f^{-4}$ law is based on quenching of energy input from the air because the wind profile is affected by the water waves.

## 2. Quasilinear equations for wind-generated water waves

It is the purpose of this section to obtain a closed set of quasilinear equations for wind-generated water waves. These equations describe the energy transfer from a two-dimensional parallel shear flow in an inviscid incompressible fluid (air) to the water waves and reversely the effect of the water waves on the air flow.

Turbulent stresses will be neglected in this treatment, although their effect on the generation of water waves will be discussed at the end of this section.

The basic equations for air and water then read

$$\left.\begin{aligned} &\text{air} \quad \nabla.\mathbf{u} = 0, \quad \frac{d}{dt}\mathbf{u} = -\frac{\nabla p}{\rho_\mathrm{a}} + \mathbf{g} \quad (z > \eta(x,t)); \\ &\text{water} \quad \nabla.\mathbf{u} = 0, \quad \frac{d}{dt}\mathbf{u} = -\frac{\nabla p}{\rho_\mathrm{w}} + \mathbf{g} \quad (z < \eta(x,t)); \end{aligned}\right\} \tag{1}$$

where $z = \eta(x,t)$ is the equation for the interface between air and water, $\mathbf{u}$ is the velocity, $\rho$ the density, $p$ the pressure, $\mathbf{g}$ the acceleration due to gravity and

$$d/dt = \partial/\partial t + \mathbf{u}.\nabla.$$

The subscripts a and w denote air and water respectively.

The interface motion is defined by (kinematic condition)

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\eta = w, \quad z = \eta(x,t), \tag{2}$$

where $u$ and $w$ are the $x$- and $z$-components of the velocity $\mathbf{u}$.

The boundary condition on the interface reads

$$p_\mathrm{a} = p_\mathrm{w}, \quad z = \eta(x,t). \tag{3}$$

The water motion is assumed to be irrotational. For simplicity the water depth is infinite so that $w(-\infty) = 0$. Miles (1957), using the set of equations (1)–(3), has investigated the linear stability of the following equilibrium state:

$$\left.\begin{aligned} &\rho_\mathrm{a} = \text{const.}, \quad \mathbf{u}_0 = U_0(z)\,\hat{\mathbf{e}}_x, \quad \frac{\partial}{\partial z}p_{\mathrm{a}0} = -\rho_\mathrm{a} g \quad (z > 0); \\ &\rho_\mathrm{w} = \text{const.}, \quad \mathbf{u}_0 = 0, \quad \frac{\partial}{\partial z}p_{\mathrm{w}0} = -\rho_\mathrm{w} g \quad (z < 0); \end{aligned}\right. \tag{4}$$

where $\hat{\mathbf{e}}_x$ is a unit vector in the $x$-direction.

For perturbations that propagate in the $x$-direction and vanish for $|z| \to \infty$, the principal result Miles found was that there is energy transfer from the air flow to the water waves if the curvature of the velocity profile at that point in the profile where the air speed is equal to the wave speed is negative.

Miles' treatment is, however, only valid on a short time scale, since in the course of time nonlinear effects may become important owing to the growth of the water waves. The energy transfer from the air to the water waves gives rise, for example, to a modification of the velocity profile which in turn affects the growth of the water waves, possibly quenching the instability.

In this paper we are primarily interested in the effect of the water waves on the air flow. In addition, we are concerned with a statistical description of the interaction of water waves and air, i.e. we consider the evolution in time of ensemble-averages of quantities like the energy density. To this end the nonlinear set of equations (1)–(3) is solved iteratively by means of a systematic expansion of the relevant quantities in powers of a small parameter. Finally, the appropriate averages are taken to obtain equations for the averaged quantities. Thus, we consider a weakly nonlinear system for which the random-phase approximation is assumed to be valid (Hasselmann 1967; Davidson 1972).

The small parameter in the expansion of quantities like energy density is taken to be the ratio of air density to water density. The argument for this choice is as follows. From Stewart (1967), where it is observed that a substantial amount of energy is contained in the water waves, it may be inferred that

$$\rho_w\langle w^2\rangle = O(\rho_a U_0^2). \tag{5}$$

Here, the angle brackets denote ensemble average, and $U_0$ is the mean air speed. Consequently, $\langle w^2\rangle = O(\epsilon^2 U^2)$, where $\epsilon^2$ is given by

$$\epsilon^2 = \rho_a/\rho_w. \tag{6}$$

It is therefore tempting to expand the elevation, the velocity and the pressure in powers of $\epsilon$. Thus

$$\left.\begin{aligned} \eta &= \sum_{l=1} \epsilon^l \eta_l, \quad \mathbf{u}_w = \sum_{l=1} \epsilon^l \mathbf{u}_{w,l}, \quad p_w = \sum_{l=0} \epsilon^l p_{w,l}, \\ \mathbf{u}_a &= \mathbf{U} + \sum_{l=1} \epsilon^l \mathbf{u}_{a,l}, \quad p_a = \epsilon^2 \sum_{l=0} \epsilon^l p_{a,l}. \end{aligned}\right\} \tag{7}$$

We remark that the series for the air pressure starts with a term $O(\epsilon^2)$ since

$$p_a = O(\rho_a u_a^2).$$

A straightforward iterative solution of a set of equations may, however, give rise to secular terms (in time, for example) in the series solution (Davidson 1972). For this reason we introduce different time scales such that there is sufficient freedom to prevent secularity. To that end it is sufficient to assume that averaged quantities, such as $\langle w^2\rangle$, depend on $\tau_0 = t$, $\tau_2 = \epsilon^2 t, \ldots$ (cf. Davidson 1972). Hence, for example,

$$\frac{\partial}{\partial t}\langle w^2\rangle = \sum_{l=0} \epsilon^{2l} \frac{\partial}{\partial \tau_{2l}}\langle w^2\rangle. \tag{8}$$

The $\tau_0$ scale takes account of the relatively rapid wave oscillations, while growth of the waves due to atmospheric input occurs on the $\tau_2$ scale, since this energy input is proportional to $\epsilon^2 = \rho_a/\rho_w$ (Miles 1957). It is the condition resulting from the elimination of secular behaviour on the short time scale $\tau_0$ which gives us the slow time dependence of the wave energy density and the mean air speed $U$.

In § 2.1 we discuss the iterative solution of the set of nonlinear equations for water, whereas the equations for air are treated in § 2.2.

### 2.1. *Iterative solution for water waves*

Since the water motion is irrotational a velocity potential $\phi$ is introduced according to

$$\mathbf{u} = \nabla\phi, \tag{9}$$

hence from incompressibility we obtain the potential equation

$$\Delta\phi = 0, \tag{10}$$

where $\Delta = \partial^2/\partial z^2 + \partial^2/\partial x^2$. With the boundary condition $\phi \to 0$, $z \to -\infty$, the solution may be written as

$$\phi = \int dk\, \hat{\phi}(k,t) \exp(ikx + |k|z). \tag{11}$$

In addition, we write for $\eta$ and $p_a$

$$\eta = \int dk\, \hat{\eta}(k,t) \exp(ikx), \quad p_a = \int dk\, \hat{p}_a(k,z,t) \exp ikx. \tag{12}$$

By means of (11) and (12) we obtain from (1)–(3) the following equation for the Fourier transform of $\eta$:

$$\frac{\partial^2}{\partial t^2}\hat{\eta}+\sigma^2\hat{\eta} = -|k|\hat{p}_a/\rho_w+\mathrm{NL} \quad (z=0), \tag{13}$$

where $\sigma^2 = g|k|$ and NL represents all the nonlinear terms (these are $O(\epsilon^2)$). Using the series given in (7) and the multiple-time scale expansion (8), we obtain to lowest order in $\epsilon$

$$\frac{\partial^2}{\partial \tau_0^2}\hat{\eta}_1+\sigma^2\hat{\eta}_1 = 0, \tag{14}$$

i.e. on the time scale $\tau_0$ we deal with free waves in the absence of air. One may proceed in this fashion to obtain the effect of the nonlinearities and the atmospheric input on the evolution in time of $\hat{\eta}_1$. See for this e.g. Hasselmann (1967), who has obtained the equation for the slow time evolution of the energy density $F(k)$, which is normalized according to

$$\int F(k)\,dk = \rho_w g\langle\eta^2\rangle = \mathscr{E}. \tag{15}$$

Here, $\mathscr{E}$ is the wave energy, and the brackets denote an ensemble average. To that end we write

$$F(k) = \epsilon^2 F_2+\epsilon^4 F_4+\dots .$$

The slow time dependence of $F_2$ to order $\epsilon^4$ is then given by

$$\frac{\partial}{\partial \tau_2}F_2 = -\frac{1}{2}\left(\hat{p}^*_{a,1}\frac{\partial}{\partial \tau_0}\hat{\eta}_1+\mathrm{c.c.}\right), \tag{16}$$

where the asterisks and c.c. denote complex conjugation.

Equation (16) results from the requirement that there be no secularity in $F_4$ on the $\tau_0$ scale. Three-wave interactions do not contribute to the slow time evolution of $F_2$ because for gravity waves they are not resonant. Resonant four-wave interactions are $O(\epsilon^6)$. Their effect may still be important if the atmospheric input term is numerically small. We return to this matter at the end of this section.

According to (16) the energy density $F_2$ changes in time owing to linear effects only. For this reason the term quasilinear approximation is used (Drummond & Pines 1962; Bernstein & Engelmann 1966; Davidson 1972).

In § 2.2 the pressure input term will be determined. We note in advance that we are especially interested in the $\tau_2$ dependence of this energy transfer.

### 2.2. *Treatment of the equations for air*

Since we do not allow for damping of both waves and turbulence, the dynamical equations for air read

$$\frac{d}{dt}\mathbf{u}_a = -\frac{\nabla p_a}{\rho_a}+\mathbf{g}, \quad \nabla.\mathbf{u}_a = 0, \tag{17}$$

and for simplicity we write

$$\mathbf{u}_a = \mathbf{U}+\delta\mathbf{u}, \quad p_a = P_0+\delta p_a, \tag{18}$$

where $\delta\mathbf{u}$ and $\delta p_a$ represent the fluctuating part of the series given in (7) (hence

$\langle\delta\mathbf{u}\rangle = \langle\delta p_a\rangle = 0$), while $\mathbf{U}$ and $P_0$ denote the steady-state part. By means of (18) we obtain from (17) an equation for $\mathbf{U}$ and for the fluctuation $\delta\mathbf{u}$

$$\frac{\partial}{\partial t}U_\beta + \frac{\partial}{\partial x_\alpha}\langle\delta u_\alpha\,\delta u_\beta\rangle = -\frac{1}{\rho_a}\frac{\partial}{\partial x_\beta}\langle P_0\rangle + g_\beta, \tag{19a}$$

$$\frac{\partial}{\partial t}\delta u_\beta + \frac{\partial}{\partial x_\alpha}(U_\alpha\,\delta u_\beta + \delta u_\alpha\,U_\beta) = -\frac{1}{\rho_a}\frac{\partial}{\partial x_\beta}\delta p_a + \frac{\partial}{\partial x_\alpha}T_{\alpha\beta}, \tag{19b}$$

where $T_{\alpha\beta} = \delta u_\alpha\,\delta u_\beta - \langle\delta u_\alpha\,\delta u_\beta\rangle$. Here, the subscripts $\alpha$ and $\beta$ denote the various components of the vector quantities $\mathbf{U}$, $\delta\mathbf{u}$ and $\mathbf{g}$, and the summation convention is assumed. Finally,

$$\frac{\partial}{\partial x_\alpha}\delta u_\alpha = 0. \tag{20}$$

Elimination of the pressure fluctuation $\delta p_a$ from (19) and (20) gives an equation for the vertical component of $\delta\mathbf{u}$:

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\Delta - U''\frac{\partial}{\partial x}\right]\delta w = \frac{\partial}{\partial x}(\nabla\times\nabla\,.\,\mathbf{T}), \tag{21}$$

where the prime denotes differentiation with respect to $z$. In obtaining (21) we have assumed that $\mathbf{U}$ points in the $x$-direction and is a function of $t$ and $z$ only.

We restrict our attention to the set (19$a$), (20) and (21). Equation (19$a$) describes the rate of change of the steady-state velocity $U$ due to the wave-induced stresses $\langle\delta u_\alpha\,\delta u_\beta\rangle$. To calculate $\langle\delta u_\alpha\,\delta u_\beta\rangle$ we need to solve (21). This may be done iteratively because $\delta w$ is assumed to be small.

In agreement with (7) we expand $\langle\delta u_\alpha\,\delta u_\beta\rangle$ in powers of $\epsilon^2$,

$$\langle\delta u_\alpha\,\delta u_\beta\rangle = \epsilon^2\langle\delta u_\alpha\,\delta u_\beta\rangle_2 + \dots, \tag{22}$$

while we also expand $U$ according to

$$U = U_0 + \epsilon^2 U_2 + \dots\,. \tag{23}$$

Substitution of (8), (22) and (23) in the $x$-component of (19$a$) then gives the hierarchy of equations

$$\frac{\partial}{\partial\tau_0}U_0 = 0, \quad \frac{\partial}{\partial\tau_0}U_2 = -\frac{\partial}{\partial\tau_2}U_0 - \frac{\partial}{\partial x_\alpha}\langle\delta u_\alpha\,\delta u\rangle_2. \tag{24}$$

The first equation of (24) tells us that $U_0$ is independent of $\tau_0$. Integration of the second equation of (24) with respect to time gives

$$U_2(\tau_0) - U_2(0) = -\tau_0\left[\frac{\partial}{\partial\tau_2}U_0 + \frac{\partial}{\partial x_\alpha}\langle\delta u_\alpha\,\delta u\rangle_2\right], \tag{25}$$

where, as will be shown, $\langle\delta u_\alpha\,\delta u\rangle_2$ is independent of $\tau_0$. In order to avoid secularity in $U_2$ on the $\tau_0$ scale we require the vanishing of the right-hand side of (25):

$$\frac{\partial}{\partial\tau_2}U_0 = -\frac{\partial}{\partial x_\alpha}\langle\delta u_\alpha\,\delta u\rangle_2, \tag{26}$$

resulting in an equation for the $\tau_2$ dependence of $U_0$. Although this result may seem trivial (cf. (19$a$)), it is not. To obtain the $\tau_2$ dependence of $U_0$ we only need the wave-induced stress to lowest significant order.

To calculate $\langle \delta u_\alpha \delta u \rangle_2$ we solve (21). To lowest order we find

$$\left(\frac{\partial}{\partial \tau_0} + U_0 \frac{\partial}{\partial x}\right) \Delta \delta w_1 = U_0'' \frac{\partial}{\partial x} \delta w_1, \tag{27}$$

i.e. $\delta w_1$ satisfies the well-known Rayleigh equation.

The boundary conditions for $\delta w_1$ follow from the requirements that the interface shall remain a streamline and that the fluctuation $\delta w$ vanishes at infinity. To lowest order we therefore obtain

$$\delta w_1(z = 0) = \frac{\partial}{\partial z}\phi(0); \quad \delta w_1 \to 0 \quad \text{as} \quad z \to \infty. \tag{28}$$

By means of the lowest-order solution found in § 2.1 the first boundary condition is given by

$$\delta w_1(z = 0) = -\int_0^\infty dk\, \sigma(k) \{i\hat{\eta}_1 e^{i(kx - \sigma\tau_0)} + i\hat{\eta}_2 e^{i(kx + \sigma\tau_0)}\} + \text{c.c.} \tag{29}$$

Apparently, the air at $z = 0$ is forced to oscillate in the manner prescribed by (29). As suggested by this boundary condition we therefore try the solution

$$\delta w_1 = -\int_0^\infty dk\, \sigma(k) \{i\hat{\eta}_1 \chi_1 e^{i(kx - \sigma\tau_0)} + i\hat{\eta}_2 \chi_2 e^{i(kx + \sigma\tau_0)} + \text{c.c.}\}, \tag{30}$$

to obtain the following problem for $\chi_1$:

$$(W\Delta - W'')\chi_1 = 0; \quad \chi_1(0) = 1, \quad \chi_1(\infty) = 0; \tag{31}$$

where $W = \sigma - kU_0$, and $\Delta = \partial^2/\partial z^2 - k^2$. Since $\sigma$ and $k$ as well as $U_0$ are positive, resonance of the wave with the air flow is possible only for the $\chi_1$-component of $\delta w_1$. From now on, we therefore omit the contributions of the waves propagating to the left, and we drop the subscript 1.

From incompressibility we have

$$\delta u_1 = \int_0^\infty dk \left\{\frac{\sigma}{k} \hat{\eta}_1 \frac{\partial}{\partial z} \chi e^{i(kx - \sigma\tau_0)} + \text{c.c.}\right\},$$

to obtain

$$\langle \delta u\, \delta w \rangle_2 = -i \int_0^\infty dk \frac{F_2}{\rho_w} \left(\chi \frac{\partial}{\partial z} \chi^* - \chi^* \frac{\partial}{\partial z} \chi\right). \tag{32}$$

Now,

$$\frac{\partial}{\partial x_\alpha} \langle \delta u_\alpha \delta u \rangle_2 = \frac{\partial}{\partial z} \langle \delta u\, \delta w \rangle_2,$$

since $\partial \langle \delta u^2 \rangle_2 / \partial x = 0$; hence (26) may be written as

$$\frac{\partial}{\partial \tau_2} U_0 = +i \int_0^\infty dk \frac{F_2(k, \tau_2)}{\rho_w} \left(\chi \frac{\partial^2}{\partial z^2} \chi^* - \chi^* \frac{\partial^2}{\partial z^2} \chi\right). \tag{33}$$

Finally, the term between brackets may be evaluated by means of the Rayleigh equation (31) to obtain one of our main results

$$\frac{\partial}{\partial \tau_2} U_0 = -2\pi \int_0^\infty dk \frac{F_2(k, \tau_2)}{\rho_w} |\chi|^2 W'' \delta(W), \tag{34}$$

where $\delta(W)$ is a delta-function. Performing the integration over $k$, we obtain an equation of the diffusion type:

$$\frac{\partial}{\partial \tau_2} U_0 = 2\pi\sigma^2 \frac{E_2(\sigma, \tau_2)}{\rho_w g} |\chi|^2 \frac{\partial^2}{\partial z^2} U_0, \quad \sigma = \frac{g}{U_0}. \tag{35}$$

Here we introduced the energy density $E_2(\sigma, \tau_2)$ according to

$$\int dk\, F_2(k, \tau_2) = \int E_2(\sigma, \tau_2)\, d\sigma. \tag{36}$$

Equation (35) tells us that the air flow at a certain height $z$ changes with time owing to resonant interaction of a water wave with frequency $\sigma = g/U_0(z)$. Hence, in this fashion there is possibly an energy transfer from the air flow $U_0$ to the water waves, thus giving a rate of change of the spectrum $E_2$ (cf. (16))

$$\frac{\partial}{\partial \tau_2} E_2 = -\frac{\sigma}{g}\left(\hat{p}^*_{a,1} \frac{\partial}{\partial \tau_0} \hat{\eta}_1 + \text{c.c.}\right).$$

Using the $z$-component of (19*b*), we can write the air-pressure fluctuation $\hat{p}_{a,1}$ as

$$\hat{p}_{a,1} = -i\rho_w \int_0^\infty dz\, W\hat{w}(z=0); \tag{37}$$

hence, with $\hat{w} = -i\sigma\hat{\eta}_1\chi$ we obtain

$$\frac{\partial}{\partial \tau_2} E_2 = \frac{\sigma^3}{g} \rho_w |\hat{\eta}_1|^2 \left[ i \int_0^\infty dz\, W\chi + \text{c.c.} \right].$$

Then, by means of the Rayleigh equation (31) we obtain the well-known result (Miles 1957)

$$\frac{\partial}{\partial \tau_2} E_2 = -\frac{\pi\sigma}{k} |\chi_c|^2 \frac{W''_c}{W'_c} E_2, \tag{38}$$

where the subscript c refers to evaluation at the critical height ($U_0 = g/\sigma$).

To summarize our results, we obtain the following set of quasilinear equations for the generation of water waves by the wind:

$$\left.\begin{aligned} &\frac{\partial}{\partial t} E = -\pi\epsilon\sigma \frac{|\chi_c|^2}{k} \frac{W''_c}{W'_c} E, \quad W = \sigma - kU_0, \\ &\frac{\partial}{\partial t} U_0 = D(z, U_0) \frac{\partial^2}{\partial z^2} U_0, \quad D = 2\pi\sigma^2 \frac{E(\sigma, t)}{\rho_w g} |\chi|^2, \\ &W\Delta\chi = W''\chi, \quad \chi(0) = 1, \quad \chi(\infty) = 0, \end{aligned}\right\} \tag{39}$$

where we have returned to the original variables ($t = \tau_2/\epsilon^2, E = \epsilon^2 E_2$).

From the first equation of (39) we obtain the well-known result that only those waves are unstable for which the curvature $U''_0$ of the wind profile at the critical height is negative (this is, for example, the case for a logarithmic velocity profile). The growth rate of the waves is, however, a function of time, as the wind profile depends on time according to the diffusion equation for $U_0$, possibly quenching the instability for large $t$. This topic will be discussed in more detail in § 3.

It should be noted that Lighthill (1962), who discussed the physical interpretation of Miles' theory of wave generation by the wind, obtained a similar result regarding the effect of a single wave on the wind profile. He did not realize however that the wind profile $U_0$ may be a slowly varying function of time.

In addition, Fabrikant (1976) obtained a similar set of quasilinear equations, although along different lines. Here we once again emphasize that, by means of the multiple-time-scale technique, equations for the slowly varying quantities $F$ and $U_0$ are obtained from the requirement that there be no secularity of the second-order quantities (e.g. $U_2$) on the time scale $\tau_0$. This renders the series solution, given in (7), uniformly valid up to $t = O(\epsilon^{-2})$.

Finally, two objections may be raised against the validity of the quasilinear theory of wind-generated water waves. The first objection is related to the effect of turbulent Reynolds stresses on the growth of the water waves and on the wind profile. Including the turbulence of the air flow one arrives at the following diffusion equation for $U_0$:

$$\frac{\partial}{\partial t} U_0 = D_{\mathrm{w}} \frac{\partial^2}{\partial z^2} U_0 + \frac{\partial}{\partial z} A(z) \frac{\partial}{\partial z} U_0, \tag{40}$$

where $D_{\mathrm{w}}$ is the diffusion coefficient related to the effect of the water waves on the air flow (in the absence of turbulence it is given by $D$ in (39)), while $A(z)$ is the eddy viscosity of the air.

Clearly, air turbulence may give rise to a broadening of the critical layer (say, the $\delta$-function in (34) is replaced by its Lorentzian counterpart) giving a smoothing of the effect of the water waves on the wind profile.

Also, the effect of eddy viscosity (i.e. the second term on the right-hand side of (40)) is clear. If no water waves are present the well-known logarithmic wind profile is obtained in the steady state (since $A = \alpha z$).

In the presence of water waves the eddy-viscosity term is capable of maintaining this logarithmic wind profile if $D_{\mathrm{w}} \ll A$. At later stages of the wave growth, however, the effect of the waves on the wind profile may overcome eddy viscosity, especially in the layer just above the water waves.

The second objection concerns the effect of resonant four-wave interactions and dissipation on the evolution in time of the spectrum. In the derivation of the quasilinear set of equations (39) it was assumed that the growth rate of the gravity waves due to the wind was $O(\epsilon^2)$. Numerically, however, the growth rates are quite small, because, for example, the curvature in the wind profile is small. Therefore, in general, four-wave interactions and dissipation are important processes, which should also be taken in account (Hasselmann 1978).

Thus, our model (39) is not really realistic, and its results can only be compared with controlled experiments in wind–wave tunnels, where particular sets of conditions can be produced in isolation. We add to this, that it is our only purpose to investigate some properties of the coupled wind–wave system.

## 3. Some exact consequences of the quasilinear equations

Let us investigate some properties of the quasilinear theory of wind wave generation. First of all we question whether the set of equations (39) admits a steady state. To that end we derive an equation for the enstrophy. We differentiate the diffusion equation for $U_0$ with respect to $z$ to obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z} U_0\right) = \frac{\partial}{\partial z}\left[D \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} U_0\right)\right] \quad (D > 0). \tag{41}$$

We next multiply (41) by $\partial U_0/\partial z$ and integrate over $z$ with the result

$$\frac{d}{dt}\int_0^\infty \left(\frac{\partial}{\partial z}U_0\right)^2 dz = 2\int_0^\infty dz \frac{\partial}{\partial z}U_0 \frac{\partial}{\partial z}\left[D\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}U_0\right)\right]$$

$$= 2\int_0^\infty dz\left[\frac{\partial}{\partial z}\left(D\frac{\partial}{\partial z}U_0\frac{\partial^2}{\partial z^2}U_0\right) - D\left(\frac{\partial^2}{\partial z^2}U_0\right)^2\right].$$

For the boundary conditions

$$\frac{\partial^2}{\partial z^2}U_0(0) = 0, \quad \frac{\partial^2}{\partial z^2}U_0(\infty) = 0$$

the perfect derivative on the right above integrates to zero, hence

$$\frac{d}{dt}\frac{1}{2}\int_0^\infty \left(\frac{\partial}{\partial z}U_0\right)^2 dz = -\int_0^\infty dz\, D\left(\frac{\partial^2}{\partial z^2}U_0\right)^2. \tag{42}$$

This equation states that the time derivative of the enstrophy, which is bounded from below by zero, is non-positive. Hence, the system tends towards a condition where the right-hand side vanishes, which requires in the region where $D \neq 0$ that

$$\frac{\partial^2}{\partial z^2}U_0 = 0 \quad \text{as} \quad t \to \infty. \tag{43}$$

Thus, for large times the wind profile becomes linear, implying that according to (39) the growth rate of the waves vanishes. Apparently, quasilinear theory predicts a limitation of the amplitude of the initial unstable water waves for large times, i.e. the energy transfer from the air flow to the water waves is quenched.

In the following we concentrate on the asymptotic form of the spectrum of gravity waves. We should first note however that the set of quasilinear equations (39) admits an infinite set of balance equations, notably

$$\frac{d}{dt}\left[\rho_a\int_0^\infty f(U_0)\,dz + 2\int_0^\infty \frac{\sigma}{g}f'(U_0)\,E(\sigma,t)\,d\sigma\right] = 0, \tag{44}$$

where $f(U_0)$ is an arbitrary function of $U_0$ (such that the integrals exist), the prime denotes differentiation with respect to $U_0$, and in the second term of the left-hand side $U_0 = g/\sigma$.

Equation (44) may be obtained by multiplication of the diffusion equation for $U_0$ by $f'(U_0)$; then integration with respect to $z$ gives

$$\frac{d}{dt}\int_0^\infty f(U_0)\,dz = \int_0^\infty f'(U_0)\,D\frac{\partial^2}{\partial z^2}U_0\,dz.$$

In the integral on the right-hand side we next convert to an integration over $\sigma$ via $U_0 = g/\sigma$, then using the expression for $D$ and the evolution equation for the spectrum $E$ we finally arrive at the conservation law (44).

By an appropriate choice of $f'(U_0)$ we are able to express all moments of the spectrum $E$ in terms of an integral of $f(U_0)$ in $z$-space (provided of course the integrals exist). In particular, we obtain for $f'(U_0) = 1$, conservation of momentum,

$$\frac{d}{dt}\left[\rho_a\int_0^\infty U_0\,dz + \frac{2}{g}\int_0^\infty \sigma E(\sigma,t)\,d\sigma\right] = 0, \tag{45a}$$

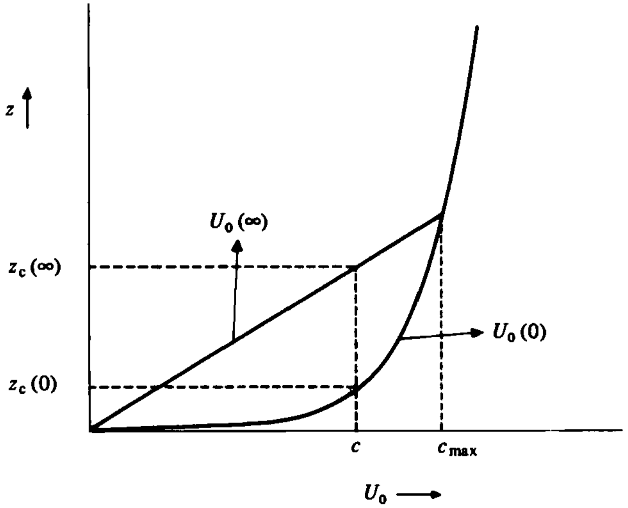


FIGURE 1. Initial and asymptotic forms of the wind profile.

and for $f'(U_0) = U_0$ conservation of mechanical energy,

$$\frac{d}{dt}\left[\int_0^\infty \tfrac{1}{2}\rho a\, U_0^2\, dz + 2\int_0^\infty E(\sigma, t)\, d\sigma\right] = 0. \tag{45b}$$

From these conservation laws we see once more that for growing waves the wind profile changes in time.

Another important feature of the quasilinear theory is that the critical height $z_c$ is a function of time, as may be inferred from figure 1. The critical height $z_c$ is obtained from the condition

$$U_0(z_c, t) = c = g/\sigma. \tag{46}$$

It is of interest to calculate $\partial z_c/\partial t$. We therefore differentiate (46) with respect to time, keeping $\sigma$ fixed, to obtain

$$\frac{\partial}{\partial t} z_c = -\frac{\partial}{\partial t} U_{0c} \bigg/ \frac{\partial}{\partial z} U_{0c}.$$

Eliminating $\partial U_{0c}/\partial t$ via the diffusion equation for $U_0$ and using the evolution equation for the spectrum the result is

$$\frac{\partial}{\partial t} z_c = \frac{2\sigma^3}{\rho_a g^2} \frac{\partial}{\partial t} E, \tag{47}$$

which gives for large $t$ the important relation

$$E(\infty) = E(0) + \frac{\rho_a g^2}{2\sigma^3}\left[z_c(\infty) - z_c(0)\right]. \tag{48}$$

Equation (48) expresses the asymptotic form of the spectrum in terms of its initial value and the change in the critical height.

We finally evaluate the asymptotic spectrum $E(\infty)$ for a logarithmic velocity profile at $t = 0$,

$$U_0(z, 0) = \frac{u_*}{\kappa} \ln\left(1 + \frac{z}{z_0}\right),$$

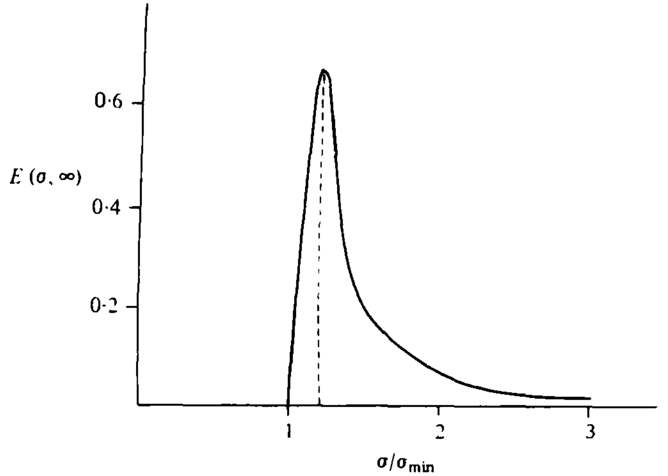


FIGURE 2. The spectrum $E(\sigma, \infty)$ as a function of $\sigma/\sigma_{\min}$ for $\sigma_* = 5\sigma_{\min}$.

where $u_*$ is the friction velocity, $\kappa$ is the von Kármán constant, $z_0 = \nu/\kappa u_*$, and $\nu$ is the kinematic viscosity of the air. In addition, we assume that for $z > z_{\max}$ the curvature of the velocity profile is so small that only water waves with phase velocities $c < c_{\max}$ (where $c_{\max} = U_0(z_{\max}, 0)$) are excited. Hence, we have a spectrum of water waves for $\sigma_{\min} < \sigma < \infty$, where $\sigma_{\min} = g/c_{\max}$.

From $U_0(z_c, 0) = g/\sigma$ one now easily obtains $z_c(0)$.

On the other hand, for large $t$ we have, according to (43), a linear velocity profile for $U_0 < c_{\max}$:

$$U_0(z, \infty) = c_{\max} z/z_{\max} \quad (U_0 < c_{\max}).$$

From the resonance condition $U_0 = g/\sigma$ the expression for $z_c(\infty)$ is obtained. Assuming that at $t = 0$ the energy of the water waves is small ($E(0) = 0$), the final result is the asymptotic spectrum $E(\infty)$, given by

$$E(\infty) = \left\{\begin{array}{ll} 0 & (\sigma < \sigma_{\min}), \\ \dfrac{\rho_a g^2}{2\sigma^3} z_{\max}\left[\dfrac{\sigma_{\min}}{\sigma} - \dfrac{1-\exp(\sigma_*/\sigma)}{1-\exp(\sigma_*/\sigma_{\min})}\right] & (\sigma > \sigma_{\min}), \end{array}\right\} \tag{49}$$

where $\sigma_* = \kappa g/u_*$ (usually, $\sigma_{\min} \ll \sigma_*$). We note that $E(\sigma, \infty)$ is zero for $\sigma = \sigma_{\min}$, in agreement with the observation that the critical height corresponding to this frequency for $t \to \infty$ equals the critical height at $t = 0$. We have checked that the solution (49) satisfies conservation of momentum and energy (cf. (45*a*, *b*)). The form of the spectrum $E(\sigma, \infty)$ is given in figure 2.

Since usually $\sigma_* \gg \sigma_{\min}$ the spectrum increases rapidly until the peak value at $\sigma = \sigma_p$ is reached, followed by a decrease of the spectrum according to

$$E(\sigma, \infty) \simeq \tfrac{1}{2}\rho_a g^2 z_{\max} \sigma_{\min} \sigma^{-4} \quad (\sigma > \sigma_p). \tag{50}$$

Hence, according to quasilinear theory the high-frequency part of the spectrum drops like $\sigma^{-4}$. It is of interest to notice that Mitsuyasu *et al.* (1980) recently found that a $\sigma^{-4}$ power law fits quite well to the high-frequency part of their observed spectra. This power law was proposed by Toba (1973, 1978). Unfortunately, however, the present theory cannot be regarded as a justification for Toba's proposal because a number of important effects have not been taken into account. These are e.g. the effect of nonlinear wave interactions and the effect of turbulence on the wind profile.

Phillips (1958) obtained by a dimensional argument a $\sigma^{-5}$ power law for the so-called saturation range. In his dimensional analysis, however, only the frequency $\sigma$ and the gravity constant $g$ were assumed to be the relevant parameters for the spectrum. On the other hand, in our treatment a 'velocity' $z_{max}\sigma_{min}$ is a relevant parameter too. While the $\sigma^{-5}$ power law is based on the limiting wave form, that is on the occurrence of sharp wave crests, the $\sigma^{-4}$ power law as obtained from quasilinear theory is based on a different physical mechanism. The amplitude of the water waves is limited because the water waves modify the wind profile in such a way that eventually there is no energy transfer from the air flow to the waves.

We mention finally that quasilinear theory predicts that the momentum flux $\tau_w$ due to the interaction between the wave perturbations and the mean-velocity profile vanishes for large times or large dimensionless fetch. This is because for large times $\partial^2 U_0/\partial z^2 \rightarrow 0$ and from (39) $\tau_w = \int dz\, D\, \partial^2 U_0/\partial z^2$. Snyder *et al.* (1981) found that the ratio $\tau_w/\tau$ of wave-supported to total momentum flux was a strong inverse function of dimensionless fetch. However, they were unable to estimate $\tau_w$ directly. Thus, there is no hard evidence from this field experiment which supports the present theory.

## 4. Summary of conclusions

In this paper we have studied the generation of water waves by wind. While the water waves grow owing to energy transfer from the air flow (if its curvature is negative), the waves in turn modify the flow in such a way that the velocity profile becomes linear for large times. In the framework of quasilinear theory the waves then do not grow any more because the energy transfer is quenched. We have also presented some exact consequences of the quasilinear theory. We have discussed especially the exact asymptotic frequency spectrum. In the high-frequency range a $\sigma^{-4}$ power law is obtained.

Experimental evidence for the effects described in this paper is hard to find. In field experiments not only the above-mentioned relaxation to equilibrium due to the interaction between water waves and wind may be relevant. Other effects, such as energy transfer due to four-wave interactions and wave dissipation are bound to affect the shape of a real wave spectrum too (Hasselmann *et al.* 1973). In addition, the effect of turbulence on the wind profile also has to be taken into account. Therefore, the best comparisons should be against laboratory work, where specific effects may be isolated. To our knowledge, measurements of the velocity profile below the critical height are not available, since the measurement problem is far from simple.

The only point we wish to make is, however, that, besides nonlinear interactions and wave dissipation, the effect of the waves on the wind profile may be important too in the evolution of the wave spectrum.

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## REFERENCES

BERNSTEIN, I. B. & ENGELMANN, F. 1966 Quasi-linear theory of plasma waves. *Phys. Fluids* **9**, 937.

DAVIDSON, R. C. 1972 *Methods in Nonlinear Plasma Theory*. Academic.

DRUMMOND, W. E. & PINES, D. 1962 Nonlinear stability of plasma oscillations. *Nucl. Fusion Suppl.* **3**, 1049.

FABRIKANT, A. L. 1976 Quasilinear theory of wind-wave generation. *Izv., Atmos. Oceanic Phys.* **12**, 524.

HASSELMANN, K. 1967 Nonlinear interactions treated by the methods of theoretical physics (with application to the generation of waves by wind). *Proc. R. Soc. Lond.* A **299**, 77.

HASSELMANN, K. 1974 On the spectral dissipation of ocean waves due to white capping. *Boundary-layer Met.* **6**, 107.

HASSELMANN, K. 1978 On the spectral energy balance and numerical prediction of ocean waves. In *Turbulent Fluxes through the Sea Surface, Wave Dynamics, and Prediction* (ed. A. Favre & K. Hasselmann), p. 531. Plenum.

HASSELMANN, K., BARNETT, T. P., BOUWS, E., CARLSON, H., CARTWRIGHT, D. E., ENKE, K., EWING, J. A., GIENAPP, H., HASSELMANN, D. E., KRUSEMAN, P., MEERBURG, A., MÜLLER, P., OLBERS, D. J., RICHTER, K., SELL, W. & WALDEN, H. 1973 Measurements of wind-wave growth and swell decay during the Joint North Sea Wave Project (JONSWAP). *Deutsche Hydrog. Z., Suppl.* A (**80**), no. 12.

LANDAU, L. 1946 On the vibrations of the electronic plasma. *J. Phys.* (*U.S.S.R*) **10**, 25.

LIGHTHILL, M. J. 1962 Physical interpretation of the mathematical theory of wave generation by wind. *J. Fluid Mech.* **14**, 385.

MILES, J. W. 1957 On the generation of surface waves by shear flows. *J. Fluid Mech.* **3**, 185.

MITSUYASU, H., TASAI, F., SUHARA, T., MIZUNO, S., OHKUSU, M., HONDA, T. & RIKUSHI, K. 1980 Observation of the power spectrum of ocean waves using a cloverleaf buoy. *J. Phys. Oceanog.* **10**, 286.

PHILLIPS, O. M. 1958 The equilibrium range in the spectrum of wind-generated water waves. *J. Fluid Mech.* **4**, 426–434.

SNYDER, R. L., DOBSON, F. W., ELLIOT, J. A. & LONG, R. B. 1981 Array measurements of atmospheric pressure fluctuations above surface gravity waves. *J. Fluid Mech.* **102**, 1.

STEWART, R. W. 1967 Mechanics of the air–sea interface. *Phys. Fluids Suppl.* **10**, S47.

TOBA, Y. 1973 Local balance in the air–sea boundary processes. III. On the spectrum of wind waves. *J. Oceanog. Soc. Japan* **29**, 209.

TOBA, Y. 1978 Stochastic form of the growth of wind waves in a single-parameter representation with physical implications. *J. Phys. Oceanog.* **8**, 494.

VEDENOV, A. A., VELIKHOV, E. P. & SAGDEEV, R. Z. 1961 Nonlinear oscillations of a rarefied plasma. *Nucl. Fusion* **1**, 182.